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A note on line broadcast in digraphs under the edge-disjoint paths mode

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Abstract

It is known that unless NP $\subset$ DTIME($n^{\log\log n}$), no polynomial-time approximation algorithm for the multicast problem can have approximation ratio less than $\Omega(\log n)$ in n-node digraphs under the edge-disjoint paths mode of the line model. In this note, we give a polynomial-time $O((\Delta_{\min} + \log n)/(\log(\Delta_{\min} + \log n)))$-approximation algorithm, where $\Delta_{\min}$ is the smallest integer k such that there exists a rooted directed tree of maximum out-degree k, spanning the considered digraph.

Keywords: Interconnection network; Broadcasting; Multicasting

1. Statement of the problem

Given a source-node s in a network, and a set D of destination-nodes, *multicasting* from s to D consists in transmitting a piece of information from s to all nodes in D using the communication facilities of the network. *Broadcasting* is a particular case of multicasting in which the destination set consists of all nodes of the network. Multicasting and broadcasting are two of the basic operations upon which network applications are frequently based nowadays. They hence gave rise to a vast literature, covering both applied and fundamental aspects of the problem (cf. [8,22] and [19], respectively).

As far as graph-theoretic aspects are concerned, the multicast problem can be expressed as follows. We are given a graph $G = (V, E)$, a node $s \in V$, and a set $D \subseteq V$. We are looking for the most efficient multicast protocol from s to D in G. The nature of the "protocol" and the measure of its "efficiency" depend on the communication model.

The *local model* assumes that transmissions proceed by synchronous *calls* between the nodes of the graph. It is assumed, moreover, that (1) a call involves exactly two neighboring nodes (locality constraint), (2) a node can participate in at most one call at a time (single-port constraint), and (3) the duration of a call is 1 (atomic constraint). A multicast protocol is then described by the list of calls placed between the nodes of the graphs. The efficiency of the protocol is measured in terms of number of *rounds*, where round t is defined as the set of all calls performed between time $t-1$ and time t, $t = 1, 2, \ldots$. This model has been intensively investigated, for both specific and arbitrary topologies (cf. [16,20] and [1,2,9–11,21,23,24], respectively).

The *line model* relaxes the locality constraint, and allows calls to be placed between non-neighboring nodes. A call is then a path in the graph, whose two extremities are the "caller" and the "callee". (Note that, as opposed to the models in [4,5], the intermediate nodes between the caller and the callee do not receive the information which just "cuts through" the routers attached to them.) Several modern technologies (e.g. single-hop WDM for optical networks) support transmissions whose costs are distance-invariant, and are hence good applications for the line model.

Two main variants of the line model have been investigated.

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The *vertex-disjoint path mode* specifies that at any given round the paths joining the two participants of every call must be pairwise vertex-disjoint (cf. [14,21] and the references therein).

This note is concerned with another variant of the line-model, called *edge-disjoint paths mode*, i.e. at any given round, the paths joining the two participants of every call must be pairwise edge-disjoint. Under this latter mode, it is known [6,12] that if the graph is undirected, then an optimal $\lceil \log|D| \rceil$-round multicast protocol can be computed in polynomial time for any instance (G,s,D), $s \in D$. However, very little is known in the case of directed graphs. It is only known that there exists a polynomial-time 2-approximation algorithm for directed trees [3] (see also [7]) where a directed tree is rooted at the source and has edges directed from the root toward the leaves.

We denote by $b_s(G,D)$ the minimum number of rounds required to perform a multicast from s to $D \subseteq V$ in $G=(V,E)$. For the sake of simplicity, we assume, w.l.o.g., that $s \in D$. If $D=V$, $b_s(G,V)$ is simplified in $b_s(G)$.

There are digraphs (e.g. the directed ring) that allow a multicast to be performed in logarithmic number of rounds. However, there are also strongly connected digraphs G for which $b_s(G)=\Omega(n)$. This is typically the case of the "Fork" digraph $\vec{F}_n$ of $n \geqslant 3$ nodes: $V(\vec{F}_n)=\{u,v,w_1,\dots,w_{n-2}\}$ and $E(\vec{F}_n)=\{(u,v),(v,w_i),(w_i,u), i=1,\dots,n-2\}$. This digraph satisfies $b_u(\vec{F}_n)=\lceil n/2 \rceil$ rounds. The large range of broadcast times in digraphs motivates the study of approximation algorithms.

An algorithm for the multicast problem is a ρ-approximation algorithm if, for any instance (G,s,D) of the problem, it returns a multicast protocol from s to D in G which completes in at most $\rho b_s(G,D)$ rounds.

Using the same construction as in [14], and the inapproximability result of Feige [13] for the Minimum Set Cover problem, on can easily show that, unless $\mathrm{NP} \subset \mathrm{DTIME}(n^{\log\log n})$, the optimal solution of the multicast problem in digraphs cannot be approximated in polynomial time within less than $\Omega(\log n)$. This fact yields the following problem:

Problem 1. Design a polynomial-time $\mathrm{O}(\log n)$-approximation algorithm for the multicast problem in n-node digraphs under the edge-disjoint paths mode of the line model.

In this note, we give a polynomial-time $\mathrm{O}((\Delta_{\min}+\log n)/(\log(\Delta_{\min}+\log n)))$-approximation algorithm, where $\Delta_{\min}$ is the smallest integer k such that there exists a rooted directed tree of maximum out-degree k, spanning the considered digraph.

2. Approximation algorithm

The single-port constraint implies that the number of informed nodes can at most double at each round. The following is folklore:

Lemma 1. *$b_s(G,D) \geqslant \lceil \log|D| \rceil$ for any digraph $G=(V,E)$, any $D \subseteq V$, and any $s \in D$.*

Now, it is a trivial observation that multicasting in Hamiltonian digraphs can be done in $\lceil \log|D| \rceil$ rounds by just using an Hamiltonian cycle. This observation applies to Eulerian digraphs as well. Actually, multicasting can also be achieved in $\lceil \log|D| \rceil$ rounds if there is an Eulerian or Hamiltonian partial subgraph of G spanning D. More generally, one can derive an approximation algorithm for the broadcast problem, based on approximated solutions for the minimum degree spanning tree problem (MDST for short). Given any digraph $G=(V,E)$, let $\Delta_{\min}$ be the smallest integer k such that there exists a rooted directed tree of maximum out-degree k, spanning G. The MDST problem is defined as follows: we are given a digraph G, and we want to compute a spanning tree of maximum out-degree $\Delta_{\min}$. A (ρ,r)-approximation algorithm for the MDST problem is an algorithm which returns a spanning tree of maximum degree at most $\mathrm{O}(\rho\Delta_{\min}+r)$.

Lemma 2 (Fürer and Raghavachari [17]). *There is a polynomial-time $(1,\log n)$-approximation algorithm for the MDST problem in digraphs.*

The previous lemma will be used in combination with the next one.

Lemma 3. *There is a polynomial-time algorithm which, given any directed tree $T=(V,E)$ rooted at r and of maximum out-degree Δ, and given any $D \subseteq V$, computes a multicast protocol from r to D performing in $\mathrm{O}((\Delta/\log\Delta)\log|D|)$ rounds.*

Proof. A multicast protocol $\mathscr{M}$ from r to D in T is constructed inductively as follows. We denote by T_x the subtree of T rooted at x. Let x be a node such that T_x contains at least $|D|/2$ destination nodes, and every subtree T_y rooted at any child y of x contains less than $|D|/2$ destination nodes. The first round is a call from r to x. If $r \neq x$, then once r has performed the $r \to x$ call, it starts multicasting in $T \setminus T_x$. Once x is informed, it spends at most Δ rounds to inform its children as follows. Let $y_1, \ldots, y_q$ be the $q \leqslant \Delta$ children of x whose subtrees contain at least one destination node, and let d_i be the number of destination nodes in T_{y_i}. Assume, w.l.o.g., that $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_q \geqslant 1$. Then x calls y_1 first, then y_2, and so on until y_q. As soon as y_i is informed, it starts multicasting in T_{y_i}.

At this point, we are left with subtrees containing at most $|D|/2$ destination nodes each, and we can apply the same strategy inductively in each subtree. Since there are $\mathrm{O}(\log |D|)$ phases of induction, and since each phase requires at most $\Delta + 1$ rounds, $\mathscr{M}$ takes at most $\mathrm{O}(\Delta \log |D|)$ rounds. Actually, the order in which x calls its children allows to save a logarithmic factor. Let $b(k)$ be the maximum, taken over all destination sets D of cardinality at most k, of the completion time of $\mathscr{M}$. We have

$$b(2k) \leqslant 1 + \max_{1 \leqslant i \leqslant \Delta} (i + b(2k/i)),$$

since the ith subtree T_{y_i} cannot contain more than $2k/i$ destination nodes. Actually, since T_{y_1} cannot contain more than k destination nodes, we have

$$b(2k) \leqslant 1 + \max_{2 \leqslant i \leqslant \Delta} (i + b(2k/i)).$$

Therefore, assuming $b(k) \leqslant \Delta/\log \Delta \log k + \alpha(k)$ with $\alpha(\cdot)$ nondecreasing yields

$$b(2k) \leqslant 1 + \left(\frac{\Delta}{\log \Delta}\right) \log 2k + \max_{2 \leqslant i \leqslant \Delta} \left(i - \frac{\Delta}{\log \Delta} \log i + \alpha(2k/i)\right)$$

and thus

$$b(2k) \leqslant 1 + \alpha(k) + \left(\frac{\Delta}{\log \Delta}\right) \log 2k + \max_{2 \leqslant i \leqslant \Delta} \left(i - \frac{\Delta}{\log \Delta} \log i\right).$$

Since $\max_{2 \leqslant i \leqslant \Delta} (i - \Delta/\log \Delta \log i) \leqslant 0$, we get

$$b(2k) \leqslant 1 + \alpha(k) + \left(\frac{\Delta}{\log \Delta}\right) \log 2k$$

that is $b(2k) \leqslant (\Delta/\log \Delta) \log 2k + \alpha(2k)$, where $\alpha(2k) = \alpha(k) + 1 = \mathrm{O}(\log k)$. Therefore, $\mathscr{M}$ completes in at most $((\Delta/\log \Delta) + \mathrm{O}(1)) \log|D|$ rounds. $\square$

Theorem 1. *Let $t(n)$ be the time-complexity of a $(1, \log n)$-approximation algorithm for the MDST problem in digraphs. Then there exists a $t(n)$-time $\mathrm{O}((\Delta_{\min} + \log n)/(\log(\Delta_{\min} + \log n)))$-approximation algorithm for the multicast problem in digraphs.*

Proof. Given $G = (V, E)$, $D \subseteq V$ and $s \in V$, we compute a tree rooted at s, spanning D and of maximum degree $\Delta \leqslant c(\Delta_{\min} + \log n)$ for some constant $c > 0$, in $t(n)$ time. Applying Lemma 3, we get a multicast protocol performing in at most $\mathrm{O}((\Delta/\log \Delta) \log|D|)$ rounds, that is at most $\mathrm{O}((\Delta_{\min} + \log n)/(\log(\Delta_{\min} + \log n)) \log |D|)$ rounds. The result then follows from Lemma 1. $\square$

Remark. From Lemma 2, $t(n)$ is polynomial.

3. Conclusion

Theorem 1 could be improved by using a Steiner tree of maximum out-degree $\Delta_{\min}(D)$ instead of $\Delta_{\min}$ where $\Delta_{\min}(D)$ is the smallest integer k such that there exists a rooted directed tree of maximum out-degree k, spanning D in G. This would, however, require to extend Lemma 2 to the Steiner version of the MDST problem in directed graphs, which seems to be non-obvious. It is actually known [14] that, unless $\mathrm{NP} \subset \mathrm{DTIME}(n^{\log \log n})$, any polynomial-time (ρ, r)-approximation algorithm for the Steiner version of the MDST problem in digraphs satisfies $\rho + r = \Omega(\log n)$.[1] For its strong relationship with Problem 1, we state the following problem explicitly:

Problem 2. Design a polynomial-time $(1, \log n)$-approximation algorithm for the Steiner version of the minimum-degree spanning tree problem in digraphs.

[1] This impossibility result holds for directed graphs only and it is known [18] that the Steiner version of the MDST problem in graphs can be approximated up to an additive factor of 1.

Finally, we refer the reader to [15] where other aspects of the multicast problem are discussed, in particular the *restricted regimen* in which only specified nodes are allowed to relay messages.

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References

[1] A. Bar-Noy, S. Guha, J. Naor, B. Schieber, Multicasting in heterogeneous networks, in: 30th ACM Symposium on Theory of Computing (STOC'98), 1998, pp. 448–453.
[2] D. Barth, P. Fraigniaud, Approximation algorithms for structured communication problems, in: Ninth ACM Symposium on Parallel Algorithms and Architectures (SPAA'97), 1997, pp. 180–188.
[3] B. Birchler, A.-H. Esfahanian, E. Torng, Toward a general theory of unicast-based multicast communication, in: 21st Workshop on Graph–Theoretic Concepts in Computer Science (WG'95), Lecture Notes in Computer Science, Vol. 1017, Springer, Berlin, 1995, pp. 237–251.
[4] C.-T. Chou, I. Gopal, Linear broadcast routing, J. Algorithms 10 (1989) 490–517.
[5] I. Cidon, I. Gopal, S. Kutten, New models and algorithms for future networks, IEEE Trans. Inform. Theory IT-41 (1995) 769–780.
[6] J. Cohen, P. Fraigniaud, J.-C. Konig, A. Raspaud, Broadcasting and multicasting in cut-through routed networks, in: 11th IEEE International Parallel Processing Symposium (IPPS'97), 1997, pp. 734–738.
[7] J. Cohen, P. Fraigniaud, M. Mitjana, Polynomial time algorithms for minimum-time broadcast in tree, Theory Comput. Systems 35 (6) (2002) 641–665.
[8] C. Diot, W. Dabbous, J. Crowcroft, Multipoint communication: a survey of protocols, functions, and mechanisms, IEEE J. Selected Areas Comm. 15 (3) (1997) 277–290.
[9] M. Elkin, G. Kortsarz, Combinatorial logarithmic approximation algorithm for directed telephone broadcast problem, in: 34th ACM Symposium on Theory of Computing (STOC), 2002.
[10] M. Elkin, G. Kortsarz, Sublogarithmic approximation for telephone multicast: path out of jungle, in: 14th Annual ACM-SIAM Symp. on Discrete Algorithms (SODA), 2003, pp. 76–85.
[11] M. Elkin, G. Kortsarz, Approximation algorithm for directed telephone multicast problem, in: 30th International Colloquium on Automata, Languages and Programming (ICALP), 2003, pp. 212–223.
[12] A. Farley, Minimum-time line broadcast networks, Networks 10 (1980) 59–70.
[13] U. Feige, A threshold of $\ln n$ for approximating set cover, J. ACM 45 (1998) 634–652.
[14] P. Fraigniaud, Approximation algorithms for minimum-time broadcast under the vertex-disjoint paths mode, in: Ninth Annual European Symposium on Algorithms (ESA'01), Lecture Notes in Computer Science, Vol. 2161, Springer, Berlin, 2001, pp. 440–451.
[15] P. Fraigniaud, Minimum-time broadcast under the edge-disjoint paths mode, in: Second International Conference on Fun with Algorithms (FUN'01), Carleton Scientific, 2001, pp. 133–148.
[16] P. Fraigniaud, E. Lazard, Methods and problems of communication in usual networks, Discrete Appl. Math. 53 (1994) 79–133.
[17] M. Fürer, B. Raghavachari, Approximating the minimum degree spanning tree within one from the optimal degree, in: Third Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'92), 1992, pp. 317–324.
[18] M. Fürer, B. Raghavachari, Approximating the minimum-degree steiner tree to within one of optimal, J. Algorithms 17 (1994) 409–423.
[19] S.M. Hedetniemi, S.T. Hedetniemi, A. Liestman, A survey of gossiping and broadcasting in communication networks, Networks 18 (1986) 319–349.
[20] J. Hromković, R. Klasing, B. Monien, R. Peine, Dissemination of information in interconnection networks (broadcasting and gossiping), in: Ding-Zhu Du, D. Frank Hsu (Eds.), Combinatorial Network Theory, Kluwer Academic, New York, 1995, pp. 125–212.
[21] G. Kortsarz, D. Peleg, Approximation algorithms for minimum time broadcast, SIAM J. Discrete Methods 8 (1995) 401–427.
[22] W. Mostafa, M. Singhal, A taxonomy of multicast protocols for internet applications, Comput. Comm. 20 (1998) 1448–1457.
[23] R. Ravi, Rapid rumor ramification: approximating the minimum broadcast time, in: 35th IEEE Symposium on Foundations of Computer Science, 1994, pp. 202–213.
[24] C. Schindelhauer, On the inapproximability of broadcasting time, in: Third International Workshop on Approximation Algorithms for Combinatorial Optimization Problems (APPROX), Lecture Notes in Computer Science, Springer, Berlin, 2000.